Mechanisms for colour confinement.

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I. INTRODUCTION

There exists by now considerable experimental evidence that Quantum Chromodynamics (QCD) is the theory of strong interactions.

The QCD Lagrangean

$$\mathcal{L} = -\frac{1}{4}\vec{G}_{\mu\nu}\vec{G}^{\mu\nu} + \sum_f \bar{\psi}_f \left(i\slashed{D} - m_f\right)\psi_f \tag{1}$$

has the simplicity and the beauty of a fundamental theory.

The notation in (1) is standard. QCD is a gauge theory with gauge group $SU(3)$, coupled to the quark fields ψ_f which belong to the fundamental representation $\{3\}$. f indicates flavour species (u, d, c, s, t, b).

$\vec{G}_{\mu\nu}$ is the field strength tensor, belongs to the adjoint representation $\{8\}$, and has 8 colour components $G^a_{\mu\nu}$.

In terms of the gauge field A^a_μ

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \tag{2}$$

f^{abc} are the structure constants of the gauge group, g is the coupling constant, and $\alpha_s = g^2/4\pi$. $\vec{G}_{\mu\nu}\vec{G}^{\mu\nu}$ in (1) is a short notation for $\sum_{a=1}^{8} G^a_{\mu\nu}G^a_{\mu\nu}$,

$$D_\mu = \partial_\mu - ig\vec{T}\vec{A}_\mu \tag{3}$$

is the covariant derivative. T^a are the generators in the fundamental representation

$$\left[T^a, T^b\right] = if^{abc}T^c \tag{4}$$

with normalization

$$\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab} \tag{5}$$

An alternative notation that we will use is

$$A_\mu \equiv \vec{A}_\mu \cdot \vec{T} \equiv \sum_a A^a_\mu T^a \tag{6}$$

and

$$G_{\mu\nu} = \vec{G}_{\mu\nu} \cdot \vec{T} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right] \tag{7}$$

In this notation

$$-\frac{1}{4}\vec{G}_{\mu\nu}\vec{G}_{\mu\nu} = -\frac{1}{2}\mathrm{Tr}\left(G_{\mu\nu}G^{\mu\nu}\right) \tag{8}$$

Tr is the trace on colour indices.

Most of the evidence for the validity of QCD is obtained by probing short distances. A well known property of the theory is indeed asymptotic freedom [1,2]: the effective coupling

constant vanishes logaritmically at short distances, or at large momentum transfers. The rate of the decay $\pi^0 \to \gamma\gamma$, which is related to triangle anomaly, tell us that the number of colours of the quarks is 3. The structure functions of inclusive lepton - hadron scattering in the deep inelastic region are consistent with the scale evolution of the coupling constant governed by the perturbative beta funcion [1–3]

$$\frac{\mathrm{d}\alpha_s(\mu)}{\mathrm{d}\ln\mu} = -\frac{\beta_0}{2\pi}\alpha_s^2(\mu) - \frac{\beta_1}{(2\pi)^2}\alpha_s(\mu)^3 + \dots \tag{9}$$

with

$$\beta_0 = 11 - \frac{2}{3}N_f \tag{10}$$

$$\beta_1 = 51 - \frac{19}{3}N_f \tag{11}$$

N_f is the number of flavours with $m_f \ll \mu$. The solution of (9) at one loop is

$$\alpha_s(\mu^2) = \frac{\alpha_s(\Lambda^2)}{1 + \beta_0\alpha_s \ln\dfrac{\mu^2}{\Lambda^2}} \tag{12}$$

At large distances the coupling constant becomes large (infrared slavery), and the perturbative expansion sensless.

In addition to that the perturbative series is intrinsecally divergent. One could think of it as an asymptotic expansion of some funtion of the coupling constant. Usually the existence of such a function is proved by resummation techniques, like Borel resummation. For QCD this is problematic [4].

Alternative methods to perturbation theory must be used to explore large distances.

A fundamental problem in QCD , which involves large distances is confinement of colour. At small distances gluons and quarks are visible as elementary constituents of hadrons. However neither a free quark nor a free gluon has ever been observed in nature. Upper limits have been established for the production of quarks in high energy collisions. I will instead quote a limit deduced in the frame of standard cosmological model [5]. Assuming that the effective mass of quarks in the primordial quark gluon plasma is of the order of a few GeV, the ratios n_q/n_p and n_q/n_γ of relic quarks and antiquarks to relic protons and photons can be computed. One finds $n_q/n_p \sim 10^{-12}$ to be compared to the experimental upper limit $n_2/n_p \leq 10^{-27}$ obtained from Millikan - like experiments [6].

In the language of field theory confinement corresponds to the statement that asymptotic states are colour singlets. If QCD is the correct theory of hadrons, then confinement of colour must be built in it, i.e., it must be derivable from the lagrangean (1). The problem is then to understand by what mechanism asymptotic states of QCD are forced to be colour singlets.

II. LATTICE QCD

The most successful non perturbative approach to QCD is the lattice formulation [7]. The idea is to regularize the theory by discretizing the Euclidean space time to a cubic lattice with periodic boundary conditions.

A generic field theory is defined in terms of the Feynman path integral

$$Z[J] = \int \mathcal{D}\,\varphi \, \exp\left[-S[\varphi] - \int J\varphi \mathrm{d}x\right] \tag{13}$$

$Z[J]$ generates all correlation functions.

The integral is performed on the field configurations $\varphi(x)$, which are a continuous infinity: it is a functional integral. The way to define a functional integral is to discretize x, and then go to the limit in which the density of discrete values goes to infinity. Lattice corresponds to a step of this discretizing procedure: on a lattice the number of integration variables is finite, the integral (13) becomes an ordinary integral and can be computed numerically, e.g. by Montecarlo methods.

I will not survey in detail the lattice formulation of QCD, but only make a few general remarks on it [8].

  i) On a lattice computations are made from first principles: the theory is simulated in its full dynamical content.

  ii) To go from lattice to continuum both the ultraviolet cut-off (the lattice spacing a) and the infrared one (the lattice size La) must be removed. Asymptotic freedom helps to solve the first problem: as the coupling constant approaches zero the lattice spacing becomes smaller and smaller in physical units and the coarse structure of the lattice less and less important. In the usual notation $\beta = 2N_c/g^2$ ($N_c = 3$ is the number of colours)

$$a(\beta) \underset{\beta \to \infty}{\simeq} \frac{1}{\Lambda_L} \exp\left(-\frac{\beta}{b_0}\right) \tag{14}$$

Λ_L is the physical scale of the lattice regularization, the analog of the usual $\Lambda_{\overline{MS}}$ of the $\overline{MS}$ scheme.

In addition at large β's scaling is governed by the perturbative beta function: this regime is called asymptotic scaling, and helps in extracting physics from numerical simulations.

To eliminate the influence of the infrared cut-off the lattice size must be larger than the correlation length.

Notice that the result of numerical simulations are regularized amplitudes, which have to be properly renormalized to get physics.

  iii) An advantage of the lattice formulation is that the theory is quantized in a gauge invariant way. The building block is the link $U_\mu(\hat{n})$, which is the parallel transport from the site $\hat{n}$ to $\hat{n} + \hat{\mu}$

$$U_\mu(\hat{n}) = \exp\left(\mathrm{i}gaA_\mu(\hat{n})\right) \tag{15}$$

The measure of the Feynman integral is the Haar measure on the gauge group $\mathrm{d}U_\mu(n)$: if the group is compact the integration is finite.

$$Z[J] = \int \prod_{n,\mu} \mathrm{d}U_\mu(n) \, \exp\left[-\beta S(U)\right] \tag{16}$$

The action can be written [7]

$$S(U) = -\sum_{n,\mu<\nu} \mathrm{Re}\left[\Pi^{\mu\nu}(n)\right] \tag{17}$$

As $a \to 0$

$$\beta S(U) \underset{a\to0}{\simeq} -\frac{1}{4} G_{\mu\nu} G_{\mu\nu} a^4 + \mathcal{O}(a^6) \tag{18}$$

$\Pi^{\mu\nu}(n)$ is the parallel transport around an elementary square of the lattice (plaquette) in the plane μ, ν.

Any other choice of the action which differs from (17) by terms $\mathcal{O}(a^6)$ is equally valid, and is expected to produce the same physics in the limit $a \to 0$. In the language of statistical mechanics these actions belong to the same class of universality.

iv) The equilibrium thermodynamics of QCD at temperature T is obtained by euclidean quantization with periodic boundary conditions in time (antiperiodic for fermions), in the limit of infinite spacial volume.

On a lattice of size N_S in the three space directions and N_T in the time direction, the above conditions imply $N_T \ll N_S$ and

$$N_T a(\beta) = \frac{1}{T} \tag{19}$$

In QCD a deconfining phase transition takes place at a temperature $T \simeq 150 - 200\mathrm{MeV}$: above such temperature hadrons melt into a plasma of quarks and gluons, and colour is deconfined.

The existence of such a transition has been established by lattice simulations [9]. By now no clear experimental evidence exists of quark gluon plasma: however many experiments are on the way [10].

III. CONFINEMENT OF COLOUR

A quantity which characterizes the long range behaviour of the force acting between a quark-antiquark pair, $Q \bar{Q}$, is the Wilson loop, $W(C)$. $W(C)$ is the trace of the parallel transport along a closed path C in space time

$$W(C) = \mathrm{Tr}\left\{\mathrm{P} \, \exp \mathrm{i} \int_C A_\mu \mathrm{d}x^\mu\right\} \tag{20}$$

P indicates ordering along the path C.

Any parallel transport from x_1 to x_2 transforms as a bilocal covariant under a gauge transformation $\Omega(x)$. If

$$P_C(x_1, x_2) = \mathrm{P}\exp\int_{Cx_1}^{x_2} \mathrm{i}A_\mu \mathrm{d}x^\mu \tag{21}$$

$$P_C(x_1, x_2) \underset{\Omega(x)}{\rightarrow} \Omega(x_1)P_\ell(x_1, x_2)\Omega^\dagger(x_2) \tag{22}$$

For a closed loop $x_1 = x_2$, and the trace being cyclic $W(C)$ is gauge invariant. Wilson's action (17) is an example of Wilson loop, with contour C the elementary plaquette.

If as C we take a rectangle of sides R in some space direction and T in the time direction, let us indicate by $W(R, T)$ the corresponding Wilson loop (fig. 1).

It can be shown that $W(R, T)$ describes the propagation in time of a $Q\,\bar{Q}$ static pair at distance R and that

$$W(R, T) \underset{T \to \infty}{\simeq} \exp\left[-TV(R)\right] \tag{23}$$

with $V(R)$ the static potential.

For a confining potential at large distances

$$V(R) = \sigma R \tag{24}$$

and

$$W(R, T) \simeq \exp\left[-\sigma T R\right] \tag{25}$$

The dependence (25) is known as area-law, and signals confinement. σ is the string tension, whose empirical value is

$$\sigma = \frac{1}{2\pi}\,\mathrm{GeV}^2 \tag{26}$$

The area law (25) is observed in numerical simulations at $\beta < \beta_c$, i.e. below the deconfining transition: above it the string tension vanishes [11].

In summary lattice simulations show that colour is confined at low temperature, and that quarks and gluons can exist as free particles above some (deconfinement) temperature.

Since the early days of QCD it was suggested [12–14] that the mechanism by which colour is confined could be "dual type II superconductivity of the vacuum". Dual means that the role of electric and magnetic quantities are interchanged with respect to ordinary supeconductors. The idea was inspired by pioneering work on dual models [15]. In ref. [15] it was argued that the force between a $Q\,\bar{Q}$ pair could be produced by field configurations analogous to Abrikosov flux tubes in superconductivity. This mechanism makes the energy proportional to the length of the tube as in (24), and explains at the same time the string like behaviour of the dual amplitude [16]. After the advent of QCD the suggestion came naturally that the flux tubes could be produced by a dual Meissner effect, squeezing the chromoelectric field acting between $Q\bar{Q}$ into Abrikosov flux tubes, in the same way as happens to magnetic field in ordinary superconductors.

Lattice simulations have demonstrated the existence of chromoelectric flux tubes between $Q\bar{Q}$ pairs [17], supporting the mechanism of confinement by dual superconductivity. A direct test of the mechanism and the understanding of its origin are of fundamental interest.

IV. BASIC SUPERCONDUCTIVITY[18]

Superconductivity is a Higgs phenomenon. The effective lagrangean [19]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + (D_\mu\varphi)^\dagger (D_\mu\varphi) - V(\varphi) \tag{27}$$

with

$$V(\varphi) = -\mu^2\varphi^\dagger\varphi + \frac{\lambda}{4}(\varphi\varphi)^2 \tag{28}$$

and

$$D_\mu = \partial_\mu - \mathrm{i}2eA_\mu \tag{29}$$

couples the electromagnetic field to a charged scalar field, which describes Cooper pairs.

$\mathcal{L}$ is invariant under $U(1)$ global rotations

$$\varphi \to \mathrm{e}^{2\mathrm{i}e\alpha}\varphi \tag{30}$$

If μ^2 in (28) is negative, $V(\varphi)$ has the shape of fig.2.

φ acquires a non vanishing *vev* Φ and the $U(1)$ symmetry is broken down to rotations by angles multiple of $2\pi/q = \pi/e$, $q = 2e$ being the charge of a pair.

It proves convenient to parametrize the field φ by its modulus ψ and its phase $\mathrm{e}^{\mathrm{i}q\theta}$

$$\varphi = \psi\,\mathrm{e}^{\mathrm{i}q\theta} \tag{31}$$

Under a $U(1)$ transformation by an angle α

$$\psi \underset{U(1)}{\to} \psi \qquad \theta \underset{U(1)}{\to} \theta + \alpha \qquad A_\mu \to A_\mu + \partial_\mu\alpha \tag{32}$$

The covariant derivative of φ becomes

$$D_\mu\varphi = \mathrm{e}^{\mathrm{i}q\theta}\left[\partial_\mu\psi + \mathrm{i}q\left(\partial_\mu\theta - A_\mu\right)\psi\right] \tag{33}$$

or, putting

$$\tilde{A}_\mu = A_\mu - \partial_\mu\theta \tag{34}$$

$$D_\mu\varphi = \mathrm{e}^{\mathrm{i}q\theta}\left[\partial_\mu - \mathrm{i}q\tilde{A}_\mu\right]\psi \tag{35}$$

$\tilde{A}_\mu$ is gauge invariant and

$$\tilde{F}_{\mu\nu} = \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu = F_{\mu\nu} \tag{36}$$

In terms of the news fields the effective lagrangean becomes

$$\mathcal{L} = -\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu} + \partial_\mu\psi\partial_\mu\psi + q^2\psi^2\tilde{A}_\mu\tilde{A}_\mu - \mu^2\psi^2 - \frac{\lambda}{4}\psi^4 \tag{37}$$

and the equation of motion for the electromagnetic field

$$\partial_\mu\tilde{F}_{\mu\nu} + q^2\psi^2\tilde{A}_\nu \tag{38}$$

Neglecting vacuum fluctuations, $\psi^2 = \bar{\psi}^2$ (its *vev*) and (38) reads

$$\partial_\mu\tilde{F}_{\mu\nu} + m^2\tilde{A}_\nu = 0 \tag{39}$$

with

$$m = \sqrt{2}\,q\bar{\psi} \tag{40}$$

The parametrization (34) for the field corresponds to the well known fact that, in the Higgs phenomenon, the phase of the Higgs field supplies the longitudinal component of the photon, when it becomes massive.

In the gauge $A_0 = 0$ a static configuration, $\partial_0\vec{A} = 0$, $\partial_0\varphi = 0$ implies $E_i = F_{0i} = 0$ and (39) for the space components reads

$$\partial_i F_{ij} + m^2\tilde{A}_j = 0$$

or

$$\vec{\nabla} \wedge \vec{H} + m^2\vec{\tilde{A}} = 0 \tag{41}$$

$\vec{j} = m^2\vec{\tilde{A}}$ is known as London current: physically it implies the existence of a steady current at zero $\vec{E}$, or, since $\rho\vec{j} = \vec{E}$, zero resistivity.

Taking the curl of both sides of (41) we obtain

$$\nabla^2\vec{H} - m^2\vec{H} = 0 \tag{42}$$

Eq.(42) means that $\vec{H}$ penetrates inside the superconductor by a length $\lambda_1 = 1/m$ and is otherwise expelled from the bulk of it: this fact is known as Meissner effect.

The key parameter, both for producing zero resistivity and Meissner effect is $\Phi = \langle|\varphi|\rangle \neq 0$ being $m^2 = 2q^2|\Phi|^2$ (40).

$\Phi \neq 0$ indicates that $U(1)$ symmetry is spontaneously broken. Indeed if the vacuum $|0\rangle$ were $U(1)$ invariant, it would be

$$U(1)|0\rangle = e^{iq_0\alpha}|0\rangle \tag{43}$$

and, for any operator $\mathcal{O}$

$$\langle 0|\mathcal{O}|0\rangle = \langle 0|U(1)^\dagger\mathcal{O}U(1)|0\rangle \tag{44}$$

If $\mathcal{O}$ carries charge q, $U^\dagger\mathcal{O}U = e^{iq\alpha}\mathcal{O}$ and eq.(44) gives

$$\langle 0|\mathcal{O}|0\rangle = e^{iq\alpha}\langle 0|\mathcal{O}|0\rangle \tag{45}$$

which implies either $q = 0$ or $\langle 0|\mathcal{O}|0\rangle = 0$. A non zero *vev* of any charged operator, like φ, implies that vacuum is not invariant under $U(1)$, which is a symmetry of $\mathcal{L}$, i.e. that $U(1)$ symmetry is spontaneously broken.

A non zero *vev* of any charged operator signals superconductivity: the effective lagrangean for that operator, due to the universal coupling to A_μ, does indeed generate a mass for the photon. The ground state of a superconductor is a superposition of states with different charges (Cooper pair condensation).

In the above equations two scales of length appear: $\lambda_2 = 1/\mu$, the inverse mass or correlation length of the Higgs field, and $\lambda_1 = 1/m$, the inverse mass of the photon or penetration depth of the magnetic field. A superconductor is called type II if $\lambda_1 \gg \lambda_2$, otherwise it is named type I. The formation of Abrikosov flux tubes at intermediate values of the external field $\vec{H}$, is energetically favorable in type II superconductors. In type I there is complete Meissner effect below some value of H_c of H, and complete penetration of it above H_c, with destruction of superconductivity.

V. MONOPOLES

A. Generalities on monopoles

The difficulty in the construction of operators with non zero magnetic charge in terms of the gauge fields A_μ, stems from the fact that monopole configurations have non trivial topology. Here we shall briefly review the definition and the classification of monopoles. For a more detailed treatment we refer to re. [20], from which most of what we say here is extracted.

The most general form of Maxwell's equation in the presence of both electric (j_μ) and magnetic (j_μ^M) current is

$$\partial_\mu F_{\mu\nu} = j_\nu \tag{46}$$
$$\partial_\mu F_{\mu\nu}^* = j_\nu^M$$

where $F_{\mu\nu}^* = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ is the dual of $F_{\mu\nu}$. Free field equations $(j_\nu = j_\nu^M = 0)$ are trivially invariant under the group of transformations

$$F_{\mu\nu} \to \cos\theta F_{\mu\nu} + \sin\theta F_{\mu\nu}^* \tag{47}$$
$$F_{\mu\nu}^* \to \cos\theta F_{\mu\nu}^* - \sin\theta F_{\mu\nu}$$

or

$$\vec{E} \to \vec{E}\cos\theta + \vec{H}\sin\theta \tag{48}$$
$$\vec{H} \to \vec{H}\cos\theta - \vec{E}\sin\theta$$

In particular this holds for $\theta = \pi/2$ or

$$\vec{E} \to \vec{H} \qquad \vec{H} \to -\vec{E} \tag{49}$$

(49) is known as duality transformation.

No particles with magnetic charges have ever been observed in nature, in spite of the many attempts to detect them. As a consequence Maxwell's equations are usually written

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{50a}$$
$$\partial_\mu F^{*\mu\nu} = 0 \tag{50b}$$

The general solution of eq.(50b) is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{51}$$

with A_μ an arbitrary vector field. For any A_μ eq.'s (50b)are identically satisfied. They are indeed known as Bianchi identities. The very possibility of introducing A_μ relies on the absence of magnetic charges, eq.'s (50b). Explicitely they read

$$\mathrm{div}\vec{H} = 0 \qquad \frac{\partial\vec{H}}{\partial t} - \vec{\nabla} \wedge \vec{E} = 0 \tag{52}$$

If a monopole exists, and we insist in describing the system in terms of A_μ, the monopole must be viewed as one end of a long, infinitely thin solenoid, bringing the magnetic flux to infinity, in order to satisfy the first of eq.(52) [21](fig. 3).

The solenoid is known as Dirac string. In order to make this string physically invisible, the parallel transport of any charged particle on any closed path around it must be equal to 1, or

$$2\pi n = e \int_C \vec{A}\mathrm{d}\vec{x} = e\Phi(H)$$

where Φ is the flux of the magnetic field across the string. On the other hand by construction $4\pi Q_M = \Phi$ hence

$$Q_M = \frac{n}{2e} \tag{53}$$

This is the celebrated Dirac quantization for the magnetic charge.

In conclusion a monopole field can still be described in terms of A_μ, provided eq.(53) is satisfied, at the price of introducing a nontrivial topology. The argument can be generalized to non abelian gauge theories and to a generic distribution of charges contained in a finite region of space. The idea is to look at the multipole expansion of the field at large distances, and to allow for a nonzero magnetic monopole component. We shall restrict for simplicity to configurations with zero electric field at some time, say $t = 0$: a sufficient condition for that is that in some gauge

$$A_0(t) = -A_0(-t), \quad \vec{A}(t) = \vec{A}(-t)$$

or

$$A_0 = 0 \qquad \frac{\partial\vec{A}}{\partial t} = 0$$

at $t = 0$. Then

$$F_{0i} = \partial_0 A_i - \partial_i A_0 - \mathrm{i}g\,[A_o, A_i] = 0$$

Let us denote by $\vec{A} \equiv (A_r, A_\theta, A_\varphi)$ the components of $\vec{A}$ in polar coordinates. A_r can be made zero by a time independent gauge transformation, which does not affect A_0. Let $\Lambda(r, \theta, \varphi)$ be the parallel transport to infinity along the radius

$$\Lambda = \mathrm{P}\exp\left(\mathrm{i}\int_r^\infty A_r(r, \theta, \varphi)\,\mathrm{d}\varphi\right) \tag{54}$$

Λ is unitary, $\Lambda^\dagger = \Lambda^{-1}$. Moreover

$$\frac{\partial\Lambda}{\partial r} + \mathrm{i}A_r\Lambda = 0 \tag{55}$$

Multiplying by $\Lambda^\dagger$ to the left (55) gives

$$A_r' = -\mathrm{i}\Lambda^\dagger\frac{\partial\Lambda}{\partial r} + \Lambda^\dagger A_r\Lambda = 0 \tag{56}$$

A_r' is in fact the gauge transformed of A_r under Λ. We will be interested at the behaviour of the field at distances $r > R$ and therefore we do not worry about singularities at $r = 0$. We are thus left with $\vec{A} = (0, A_\theta, A_\varphi)$, and we look for a configuration behaving as $1/r$ as $r \to \infty$ or

$$\vec{A} = (0, \frac{a_\theta(\theta, \varphi)}{r}, \frac{a_\varphi(\theta, \varphi)}{r})$$

Again we can make $a_\theta = 0$ by a procedure similar to the one used to make $A_r = 0$. We operate a gauge transformation, independent of t and r

$$\Lambda' = \mathrm{P}\exp\left(\mathrm{i}\int_0^\theta a_\theta(\theta, \varphi)\mathrm{d}\theta\right) \tag{57}$$

We are then left with a_φ alone and the only non vanishing component of $F^{\mu\nu}$ is $F^{\theta\varphi} = \partial_\theta a_\varphi$. The field equations, outside the space occupied by matter $(r > R)$ read

$$\partial_\mu\sqrt{g}F^{\mu\nu} + [A^\mu, \sqrt{g}F^{\mu\nu}] = 0 \tag{58}$$

g is the determinant of the metric tensor and $\sqrt{g} = 1/r^2\sin^2\theta$. For $\nu = \varphi$ eq.(58) gives

$$\partial_\theta\frac{1}{\sin\theta}\partial_\theta a_\varphi = 0 \tag{59}$$

which has the general solution

$$a_\varphi = Q(\varphi)(a + b\cos\theta)$$

If we want no singularity at the north pole $a_\varphi(0) = 0$ and

$$a_\varphi = Q(\varphi)(1 - \cos\theta) \tag{60}$$

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The equation with $\nu = \theta$ reads

$$\partial_\varphi \sqrt{g} F^{\varphi\theta} + \left[a^\varphi, \sqrt{g} F^{\varphi\theta}\right] = 0 \tag{61}$$

The term with the commutator cancels and the net result is

$$\partial_\varphi Q(\varphi) = 0 \tag{62}$$

or $Q = \text{cost}$. The non abelian monopole field is the abelian field times a constant matrix Q: in eq. (59), (61) the term with the commutator which signals the non abelian nature of the gauge group has disappeared.

By our choice the Dirac string lies along the axis $\theta = \pi$.

If we had chosen the string along the axis $\theta = 0$ we would obtain

$$a_\varphi = -Q(1 + \cos\theta)$$

The two configuration differ by a gauge transformation

$$U = e^{i2Q\varphi}$$

If we demand U to be single valued

$$\exp(i4\pi Q) = 1 \tag{63}$$

which is the Dirac quantization condition if we keep in mind that in our notation A_μ incorporates the coupling constant g. Eq.(63) gives

$$gQ_{ii} = \frac{m_i}{2} \tag{64}$$

with m_i an integer.

To summarize a monopole configuration is identified by a constant diagonal matrix Q of the algebra, with integer or half integer eigenvalues, up to a a gauge transformation. This identification is known as GNO (Goddard, Nuyts, Olive) classification [22].

In $SU(N)$ Q is traceless, and is therefore identified by $N - 1$ half integer eigenvalues: they are $N - 1$ charges corresponding to the residual $U(1)^{N-1}$ gauge invariance, under the transformations which leave a matrix Q diagonal.

For configurations containing many monopoles it can be shown that by use of gauge transformations the matrix Q can be constructed, which is the sum of the Q_i's describing the single monopoles.

What matters is the topology, specifically the homotopy structure of the $(N - 1)$ $U(1)$'s obtained by GNO construction [22]. It can also be shown that the stable monopole configurations are possible only if the first homotopy group of the gauge group, $\Pi_1(G)$, is non trivial [23], i.e. if the group G is non simply connected.

$\Pi_1(SU(N))$ is always trivial: there exist no stable classical monopole configurations unless the symmetry is broken down to a subgroup H with non trivial homotopy, like $U(1)$.

B. 't Hooft - Polyakov monopoles

A well known example of classical monopole configuration is the 't Hooft - Polyakov [24,25] monopole. The theory [26] is an $SU(2)$ gauge theory coupled to a scalar field in the adjoint representation $\vec{\varphi}$: since all fields have integer isospin, the gauge group is $SO(3)$ or $SU(2)/Z_2$. The lagrangean is

$$\mathcal{L} = \frac{1}{2}(D_\mu\vec{\varphi})(D_\mu\varphi) - \frac{1}{4}\vec{G}_{\mu\nu}\vec{G}_{\mu\nu} - V(\vec{\varphi}^2) \tag{65}$$

where $D_\mu\vec{\varphi} = \partial_\mu\vec{\varphi} - g\vec{A}_\mu \wedge \vec{\varphi}$ is the covariant derivative and

$$V(\vec{\varphi}^2) = \frac{\mu^2}{2}\vec{\varphi}^2 + \frac{\lambda}{4}(\vec{\varphi}^2)^2 \tag{66}$$

is the most general potential invariant under the gauge group and compatible with renormalizability.

If $\mu^2 < 0$, $\vec{\varphi}$ acquires a nonvanishing *vev* $\langle\vec{\varphi}\rangle$, and the symmetry group breaks down to $U(1)$, which is the group of isospin rotations around the direction of $\langle\vec{\varphi}\rangle$. An explicit static solution of the equations of motion with finite energy can then be constructed.

With a special choice of the gauge the solution is a "hedgehog":

$$A_\mu^j = -\varepsilon_{\mu jk}\frac{r_k}{r^2}A(r) \qquad \frac{\varphi^i}{\langle\vec{\varphi}\rangle} = \frac{r^i}{r}F(r) \tag{67}$$

(the upper index is an isospin index).

At large distances $A(r) \to 1$, $F(r) \to 1$. By continuity $\vec{\varphi}$ must vanish at some point, which is identified with the location of the monopole.

The solution can be gauge rotated to a gauge (unitary gauge) in which $\vec{\varphi}$ is oriented along the 3-axis $\vec{\varphi} = (0, 0, \Phi)$: this gauge transformation is regular everywhere except in the points where $\vec{\varphi} = 0$, and coincides with the construction of sect. 1. In fact the configuration (67) $A_0 = 0$, $A_r = 0$

$$A_\theta = \frac{1}{g}\vec{\sigma}(\vec{n}_\perp \wedge \vec{\nu}) \tag{68}$$

$$A_\varphi = \frac{1}{g}\left[(\vec{\nu}\cdot\vec{n})\vec{n} - \vec{\nu}\right]\vec{\sigma}$$

with $\vec{n}_\perp = (\cos\varphi, \sin\varphi, 0)$; $\vec{\nu} = (0, 0, 1)$; $\vec{n} = \vec{r}/r$. $\vec{n}_\perp \wedge \vec{\nu}$ is θ independent. Therefore the gauge transformation Λ_θ which makes $A_\theta = 0$ can be written

$$\Lambda_\theta \equiv \mathrm{P}\exp\int_0^\theta gA_\theta\mathrm{d}\theta = \exp\mathrm{i}\theta\vec{\sigma}(\vec{n}_\perp \wedge \vec{\nu})$$

It is easy to check that

$$\Lambda_\theta^\dagger\vec{n}\cdot\vec{\sigma}\Lambda_\theta = \sigma_3 \tag{69a}$$

$$\Lambda_\theta^\dagger A_\theta\Lambda_\theta + \mathrm{i}\Lambda_\theta^\dagger\partial_\theta\Lambda_\theta = 0 \tag{69b}$$

$$\Lambda_\theta^\dagger A_\varphi\Lambda\theta + \mathrm{i}\Lambda_\theta^\dagger\partial_\varphi\Lambda_\theta = \sigma_3\frac{(1 - \cos\theta)}{2} \tag{69c}$$

The gauge transformation which makes φ diagonal is called an abelian projection. The abelian projection coincides with the GNO construction. The matrix Q of eq.(64) is in this model $Q = 2\sigma_3$, corresponding to a monopole of charge 2 Dirac units.

A gauge invariant field strength $F_{\mu\nu}$ can be defined [24]

$$F_{\mu\nu} = \hat{\varphi} \cdot \vec{G}_{\mu\nu} - \frac{1}{g}\hat{\varphi} \cdot (D_\mu\hat{\varphi} \wedge D_\nu\hat{\varphi}) \tag{70}$$

whit $\hat{\varphi} = \vec{\varphi}/|\vec{\varphi}|$.

Similarly we can define

$$B_\mu = \hat{\varphi} \cdot \vec{A}_\mu \tag{71}$$

B_μ is not gauge invariant, since $\vec{A}_\mu$ is not covariant under gauge transfromations. In fact $A_\mu \to U^\dagger A_\mu U + iU^\dagger \partial_\mu U$. The identity holds

$$F_{\mu\nu} = (\partial_\mu B_\nu - \partial_\nu B_\mu) - \frac{1}{g}\hat{\varphi}(\partial_\mu\hat{\varphi} \wedge \partial_\nu\hat{\varphi}) \tag{72}$$

The two terms in eq.(72) are not separately gauge invariant; only their sum is. After abelian projection the second term drops and $F_{\mu\nu}$ is an abelian gauge field, with vector potential B_μ

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{73}$$

For the solution of the form (67) $F_{\mu\nu}$ as defined by eq.(72) obeys free Maxwell equations, except in the point where $\vec{\varphi} = 0$

$$\partial_\mu F^{\mu\nu} = 0$$

At large distances (eq.(67))

$$\vec{E} = 0 \qquad \vec{H} = \frac{1}{g}\frac{\vec{r}}{r^3}$$

$F_{\mu\nu}$ for this solution is the field of a pointlike Dirac monopole. Note that in the hedgehog gauge of (67) only the second term of eq.(72) contributes to the magnetic field.

C. The abelian projection in QCD

In QCD there is no Higgs field. However any operator $\Phi(x)$ in the adjoint representation can act as an effective Higgs field [27], and can be used to define monopoles. In what follows we shall refer to $SU(2)$ gauge group for the sake of simplicity: the extension to $SU(3)$ is trivial.

In the notation of previous section

$$\Phi(x) = \vec{\Phi}(x)\vec{\sigma} \tag{74}$$

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The abelian projection is by definition a gauge transformation which diagonalizes $\Phi(x)$: like in the 't Hooft - Polyakov's monopole configuration analized in sect. 3 ,it is singular when $\Phi(x) = 0$. Those zeros will be then world lines of monopoles.

A gauge invariant field strength

$$F_{\mu\nu} = \hat{\varphi} \cdot \vec{G}_{\mu\nu} - \frac{1}{g}\hat{\varphi} \cdot (D_\mu \hat{\varphi} \wedge D_\nu \hat{\varphi}) \tag{75}$$

can be defined, which in fact is the field of a Dirac monopole in the neighbouring of a zero. Again, putting $B_\mu = \hat{\Phi} \cdot \vec{A}_\mu$

$$F_{\mu\nu} = (\partial_\mu B_\nu - \partial_\nu B_\mu) - \frac{1}{g}\hat{\varphi}\,(\partial_\mu \hat{\varphi} \wedge \partial_\nu \hat{\varphi}) \tag{76}$$

In a hedgehog gauge ($\hat{\Phi} = \hat{r}$) the first term in (76) does not contribute to monopole charge: the second term carries a net flux of magnetic field $4\pi/g$. After abelian projection instead only the first term is different from zero.

The classical construction of GNO can now be given a quantum mechanical extension.

Indeed, in each configuration appearing in the Feynman integral, monopoles will be located in the zeros of the classical version of $\Phi(x)$. The abelian projection will identify the GNO matrix defining the monopole charges: monopole charges will be additive, the field strength beeing asymptotically abelian, and the field equations linear.

Viceversa given a classical configuration with pointlike monopoles a patching of the single monopole configurations can be performed by gauge transformations, in such a way that the monopole charges are additive: the basic fact is that monopole charge is a topological property, which can be described by homotopy, and a group product in the set of paths around the Dirac strings of the different monopoles can be defined, such that the winding numbers are additive. As a consequence the GNO matrix for the configuration will be the sum of the matrices for the single monopoles. What really matters is the charge and the location of the monopoles. Any operator in the adjoint representation which is zero in the location of the monopoles, and is diagonal with Q will identify an abelian projection which makes Q diagonal.

Defining such operator will allow to label monopoles in different classical configurations which enter the Feynman integral.

In conclusion all monopoles are $U(1)$ monopoles. A monopole species is identified by an operator in the adjoint representation. Notice that the monopole charge coupled to the magnetic field of eq.(75) is invariant under the gauge group: such charges can condense in the vacuum without breaking the gauge symmetry

Notice also that if $SU(N)$ gauge symmetry is not broken, monopoles are unstable.

D. Monopoles and confinement in QCD

Any operator $\mathcal{O}(x)$ in the adjoint representation defines a monopole species by abelian projection: of course different choices for $\mathcal{O}(x)$ will define different monopole species: the number and the location of monopoles will indeed depend on the zeros of $\mathcal{O}(x)$. Dual superconductivity means condensation of monopoles in the vacuum.

Popular choices for the abelian projection correspond to the following choices for $\mathcal{O}(x)$

1)

$$\mathcal{O}(x) = P(x) \qquad (77)$$

$P(x)$ beeing the Polyakov line

$$P(\vec{x}, x_0) = \mathrm{P} \exp\left[i \oint A_0(\vec{x}, t)\mathrm{d}t\right] \qquad (78)$$

P is the parallel transport in time on the closed path from x_0 to $+\infty$ and back from $-\infty$ to x_0. $P(x)$ transforms covariantly under the adjoint representation. On a lattice, due to periodic b.c., $P(x)$ is defined as

$$P(\vec{n}, n_0) = \prod_{i=1}^{N_T} U_0(\vec{n}, n_0 + i\nu_0)$$

N_T beeing the number of links in the time direction.

2)

$$\mathcal{O}(x) = F_{ij}$$

F_{ij} any component of the field strength tensor.

3) Maximal abelian [28]. This projection is defined on the lattice by the condition that the quantity

$$M = \sum_{n,\mu} \mathrm{Tr}\left\{ U_\mu(n)\sigma_3 U_\mu^\dagger(n)\sigma_3 \right\}$$

be maximum with respect to gauge transformation $\Omega(n)$. In formulae:

$$0 = \frac{\delta M}{\delta\Omega(m)} = \frac{\delta}{\delta\Omega(m)} \sum_{n,\mu} \left\{ \Omega(n)U_\mu\Omega^\dagger(n+1)\sigma_3\Omega(n+1)U_\mu^\dagger\Omega^\dagger(n)\sigma_3 \right\}$$

The operator $\mathcal{O}(x)$ in this case is known only in the gauge where M is maximum, but non in its covarian form. In that gauge

$$\mathcal{O}(n) = \sum_\mu U_\mu^\dagger(n)\sigma_3 U_\mu(n) + U_\mu(n-\mu)\sigma_3 U_\mu^\dagger(n-\mu)$$

The question relevant to physics is: what monopoles do condense in the QCD vacuum to produce dual superconductivity, if any.

As discussed in the introduction this question is better adressed on the lattice. From the point of view of physics a possibility is that all monopoles, defined by different abelian projections, do condense in the vacuum [27]. Is this the case?

An answer to the above question is possible if we develop a tool to directly detect dual supeconductivity. Understanding the problem in $U(1)$ gauge theory will be sufficient, since the problem reduces in any case to $U(1)$ after abelian projection.

VI. DETECTING DUAL SUPERCONDUCTIVITY

To detect dual superconductivity one can use a phenomenological approach, which consists in the detection of a London current: this approach is discussed in the lectures of D. Haymaker, in this course [29].

An alternative method, which will be discussed here, is to detect a nonvanishing *vev* of a quantity with nonzero magnetic charge. In the next section we shall construct an operator with non zero magnetic charge, which will be used as a probe of dual superconductivity.

A. $U(1)$ gauge group.

Pure $U(1)$ gauge theory in the continuum formulation is a theory of free photons. On the lattice, the building block of the theory beeing the link $U_\mu(n) = \exp(\mathrm{i}eA_\mu(n)a)$, and the action beeing the plaquette (Wilson's action) interactions exist to all orders in e. Putting $\beta = 1/e^2$ a value β_c exists, $\beta_c \simeq 1.0114(2)$ such that for $\beta > \beta_c$ the system is made of free photons. For $\beta < \beta_c$ instead the interaction is strong and Wilson loops obey the area law, which implies confinement of electric charge.

In the confined phase monopoles condense in the vacuum, which behaves as a superconductor. This has been rigorously proven in ref. [30], but only for a special form of the action (the Villain action), by showing that a magnetically charged operator exists, whose *vev* is different from zero. In ref. [31] the construction has been extended to a generic form of the action.

The basic idea is the well known formula for translation

$$\mathrm{e}^{\mathrm{i}pa}|x\rangle = |x + a\rangle \tag{79}$$

The analog of (79) for a gauge field is

$$\exp\left[\mathrm{i}\int \vec{\Pi}(\vec{x}) \cdot \vec{b}(\vec{x})\mathrm{d}^3x\right] |\vec{A}(\vec{x})\rangle = |\vec{A}(\vec{x}) + \vec{b}(\vec{x})\rangle \tag{80}$$

where $\vec{A}(\vec{x})$ play the role of q coordinates, $\vec{\Pi}(\vec{x}) = \partial\mathcal{L}/\partial\dot{\vec{A}}(\vec{x})$ play the role of conjugate momenta, and $|\vec{A}(\vec{x})\rangle$ is a state in the Schrödinger representation. In the continuum

$$\Pi^i(\vec{x}, t) = F^{0i}(\vec{x}, t) \tag{81}$$

If we choose $\vec{b}$ as the vector potential describing the field of a monopole of charge $m/2g$, with the Dirac string in the direction $\vec{n}$

$$\vec{b}(\vec{x}, \vec{y}) = -\frac{m}{2g}\frac{\vec{n} \wedge \vec{r}}{r(r - \vec{r} \cdot \vec{n})} \qquad \vec{r} = \vec{x} - \vec{y} \tag{82}$$

then

$$\mu(\vec{y}, t) = \exp\left\{\mathrm{i}\int \mathrm{d}^3x\, \vec{\Pi}(\vec{x}, t)\vec{b}(\vec{x}, \vec{y})\right\} \tag{83}$$

creates a monopole of charge m Dirac units at the site $\vec{y}$ and time t.

The magnetic charge operator is

$$Q = \int \mathrm{d}^3 x \, \vec{\nabla} \vec{H} = \int \mathrm{d}^3 x \vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{A})$$

The commutator $[Q(t), \mu(\vec{y}, t)]$ can be easily evaluated, using the basic formula

$$\left[\mathrm{e}^{ipa}, q \right] = q + a$$

giving

$$[Q(t), \mu(\vec{y}, t)] = \int \mathrm{d}^3 x \vec{\nabla} \cdot \left(\vec{\nabla} \wedge \vec{b}(\vec{x}, \vec{y}) \right) \mu(\vec{y}, t) = \frac{m}{2g} \mu(\vec{y}, t)$$

In eq.(83) the Dirac string potential has been subtracted from $\vec{b}(\vec{x}, \vec{y})$.

We will compute the *vev*

$$\tilde{\mu} = \langle 0 | \mu(\vec{y}, t) | 0 \rangle$$

as a possible probe of spontaneous symmetry breaking of dual $U(1)$, i.e. as a probe of dual superconductivity.

On the lattice $\Pi^i = \frac{1}{e} \mathrm{Im} \Pi^{0i}$ and the obvious transcription for μ is, after Wick rotation to Euclidean space,

$$\mu(\vec{n}, n_0) = \exp\left[-\beta \sum_{\vec{n}'} \mathrm{Im} \Pi^{0i}(\vec{n}', n_0) b^i(\vec{n}' - \vec{n}) \right] \tag{84}$$

b^i is the discretized version of $\vec{b}$, and the factor β comes from the $1/g$ of monopole charge and $1/g$ of the field. Of course μ depends on the gauge choice for $\vec{b}$. A better definition, which makes μ independent of the choice of the gauge for $\vec{b}$ is

$$\mu(\vec{n}, n_0) = \exp\left\{ -\beta \sum_{\vec{n}'} \mathrm{Re} \left[\Pi^{0i}(\vec{n}', n_0)(\mathrm{e}^{b^i} - 1) \right] \right\} \tag{85}$$

which coincides with the previous equation for small values of b_i. The line of sites corresponding to the location of the string must be subtracted: $\vec{n}'$ in eq.(85) runs on all sites except the string.

When computing $\langle \mu \rangle$ whith the Feynman integral

$$\langle \mu \rangle = \frac{\int \prod [\mathrm{d}U_\mu] \mathrm{e}^{-S} \mu(\vec{n}, n_0)}{\int \prod [\mathrm{d}U_\mu] \mathrm{e}^{-S}} \tag{86}$$

it can be easily shown that a gauge transfromation on $\vec{b}$, is reabsorbed by the invariance of the Haar measure of integration, if the compactified form (85) is used.

Eq.(86) coincides with the construction of ref. [30] when the action has the Villain form.

A direct measurement of $\langle \mu \rangle$ gives problems with the fluctuations: indeed μ is the exponent of a quantity which is roughly proportional to the volume, and therefore has fluctuations

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of the order $V^{1/2}$. It is a well known fact in statistical mechanics that such quantities are not gaussian-distributed, and that the width of their fluctuations does not decrease with increasing statistics: the same problem occurs with the numerical determination of the partition function. Therefore, instead of $\langle\mu\rangle$ we will measure the quantity

$$\rho(\beta) = \frac{\mathrm{d}}{\mathrm{d}\beta} \ln\langle\mu\rangle \tag{87}$$

which is the analog of the internal energy in the case of partition function. $\langle\mu\rangle$ can then be reconstructed from eq.(87), by use of the boundary condition $\langle\mu\rangle = 1$ ($\beta = 0$), obtaining

$$\langle\mu\rangle = \exp \int_0^\beta \rho(\beta')\mathrm{d}\beta' \tag{88}$$

If $\langle\mu\rangle$ has a shape like in fig.4.

The expected behaviour of ρ is depicted in fig.5.

we expect a sharp negative peak around β_c. Notice that $\langle\mu\rangle$ is an analitic function of β for a finite lattice: so it cannot be exactly zero above β_c. Only in the infinite volume limit singularities can develop in the partition function and $\langle\mu\rangle$ can be zero in the ordered phase without being identically zero.

The result of numerical simulations with Wilson action are shown in fig.6.

It is known that the transition at β_c is weak first order [32]: this means that, approaching β_c from below the correlation length increases as in a second order transition, up to a certain value $\tilde{\beta}, \beta_c$, where this increase stops. For values of $\beta < \tilde{\beta}$ a finite size analysis can be performed. $\langle\mu\rangle$ in principle depends on β, on the lattice spacing and on the lattice size L, and for a finite lattice is an analytic function of β. Since the correlation length ξ is proportional to $(\beta_c - \beta)^{-\nu}$ for $\beta \le \tilde{\beta}$ the β can be traded with ξ and ν.

Finite size scaling occours when $a/\xi \ll 1$, $a/L \ll 1$ are not relevant, and

$$\langle\mu\rangle = \Phi(L^{1/\nu}(\beta - \beta_c), L) \tag{89}$$

since $\xi^{-1/\nu} \sim (\beta_c - \beta)$.

As β approaches $\tilde{\beta}_c$ $\langle\mu\rangle$ must vanish in the limit $L \to \infty$ and

$$\rho = \frac{\mathrm{d}}{\mathrm{d}\beta} \ln\langle\mu\rangle \simeq L^{-1/\nu} \frac{\Phi'(L^{1/\nu}(\beta_c - \beta))}{\Phi} \tag{90}$$

$L^{-1/\nu}\rho$ is a universal function of $L^{1/\nu}(\beta_c - \beta)$. The quality of this scaling is shown in fig.7.

If at infinite volume $\langle\mu\rangle \simeq (\beta_c - \beta)^\delta$ the index δ and β_c can be extracted from that universal function. By a best fit we obtain

$$\nu = 0.29 \pm 0.1 \tag{91a}$$

$$\beta_c = 1.0116 \pm 0.0004 \tag{91b}$$

Monopoles do condense in the confined phase $\beta < \beta_c$. This is consistent with exact results existing for the Villain action: as explained in chapter 1 the action in the lattice is determined by requiring that it coincides with the continuum action in the limit of zero

lattice spacing $a \to 0$. This gives a great arbitrariness in terms of higher order in a: in theories like QCD which have a fixed point where $a \to 0$, those terms are expected to be unimportant. In the language of statistical mechanics they are indeed called "irrelevant", and models which differ by them are said to belong to the same class of universality. The phase transition in $U(1)$ is known to be first order, and therefore strictly speaking these arguments do not apply. Anyhow our procedure coincides with that of ref. [30] and our result is consistent with them.

The virtue of the Villain action is that it allows to perform the transformation to dual variables and to get a lower bound for $\langle \mu \rangle$ for $\beta < \beta_c$. With the generic action such a procedure is not known: our numerical method supplies to this incovenience.

A duality transformation can also be performed in supersymmetric QCD with $N = 2$, allowing to demonstrate explicitly the condensation of monopoles [33].

In what follows we shall look for condensation of $U(1)$ monopoles defined by some abelian projection: of course there we do not know the form of the effective lagrangean and we only can rely on the numerical methods.

B. Monopole condensation in $SU(2)$, $SU(3)$ gauge theories

We shall apply the procedure developed in sect.VI A for $U(1)$ gauge theory to $SU(2)$ and $SU(3)$: the extension is kind of trivial, since it consists in repeating the construction of the $U(1)$ theory to the $U(1)$'s resulting fron the abelian projection.

We shall restrict our discussion to the abelian projection of the Polyakov loop: in that case the abelian electric field strength F_{0i} defined by eq.(70) only consists of the first term. If φ is the Polyakov line, its parallel transport in the time direction $D_0\varphi = 0$ and

$$F_{0i} = \hat{\varphi} \cdot \vec{G}_{0i} \tag{92}$$

The commutation relation between the field strength operators

$$\left[F^a_{0i}(\vec{x}, t), F^b_{0k}(\vec{y}, t) \right] = -\mathrm{i}\delta^{ab} \left(\delta_{ij}\partial_k - \delta_{ik}\partial_j \right) \delta^{(3)}(\vec{x} - \vec{y}) + \mathrm{i}f^{abc} \left(A^c_k(\vec{x})\delta_{ij} - A^c_j(\vec{x})\delta_{ik} \right) \delta^{(3)}(\vec{x} - \vec{y}) \tag{93}$$

are gauge covariant, and in particular the term proportional to δ^{ab}, i.e. the commutator for $a = b$, is gauge invariant, and comes from the abelian part of the field strength. This implies that the operator constructed in analogy with the $U(1)$ operator using (79) has magnetic charge m. For $SU(2)$ gauge theory in the abelian projection which diagonalizes the Polyakov line numerical simulation around the deconfinement phase transition have been performed [34]. As explained in sect.II this is done by putting the theory on a asymmetric lattice with $N_t \ll N_s$.

Fig.8 shows the simulation on a $12^3 \times 4$ lattices: as in the $U(1)$ case ρ shows a very sharp negative peak at the deconfining phase transition.

The position of the peak agrees with the known value of the deconfining transition [10] The conclusion is that the $U(1)$ symmetry related to monopole charge conservation is spontaneously broken, and hence QCD vacuum is indeed a dual superconductor.

Similar results for the two kinds of monopoles $U(1) \times U(1)$ in the case of $SU(3)$ are shown in fig.9 [35].

VII. OUTLOOK AND FUTURE PERSPECTIVES.

We have established that QCD vacuum is a dual superconductor. More studies are needed to understand what abelian projections define monopole condensing in the vacuum, and to test 't Hooft guess that all abelian projections are equivalent [27].

Our method of analysis can also be used to study questions like the order of the deconfining transition and the possible critical exponents.

In many other numerical investigations [36,37] the number density of monopoles has been counted across the deconfining phase transition, as well as their contribution to the observed quantities: the indication is that monopoles do indeed determine the dynamics (monopole dominance). This information is complementary to our result that QCD vacuum is a superconductor: it shows that the degrees of freedom involved in this phenomenon play a dominant role in dynamics.

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Figure Captions

Fig.1 Wilson Loop

Fig.2 Spontaneous breaking of $U(1)$ in the Higgs Model.

Fig.3 Dirac Monopole.

Fig.4 Expected behaviour of $\langle\mu\rangle$ in compact $U(1)$ in 4-d.

Fig.5 Expected behaviour of ρ in compact $U(1)$ in 4-d.

Fig.6 ρ for compact $U(1)$ on a $8^3 \times 16$ lattice.

Fig.7 Test of the scaling eq.(90). Data come from lattices of different sizes.

Fig.8 $SU(2)$: ρ vs β on a lattice $12^3 \times 4$

Fig.9 $SU(3)$: ρ vs β on a lattice $12^3 \times 4$.

Fig 4

Fig. 5

Fig. 6

Fig.7

Fig. 8